Introduction to Parameterized Complexity in Scheduling

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FPT vs para-NP

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- FPT $\subseteq$ para-NP, parameter k
  - FPT: deterministic time $f(k) \times poly(n)$
  - para-NP: nondeterministic time $f(k) \times poly(n)$

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  - INPUT: a set S of variables, a boolean formula $\mathcal{F}$ of size n based on S

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  - NP-hard but FPT algorithm in time $\mathcal{O}(2^k \times n)$

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  - W[1]-complete: INDEPENDENT SET, CLIQUE
  - W[2]-complete: DOMINATING SET

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- Example: $1|chains(l_{i,j}), p_j = 1, r_j, d_j|\star$

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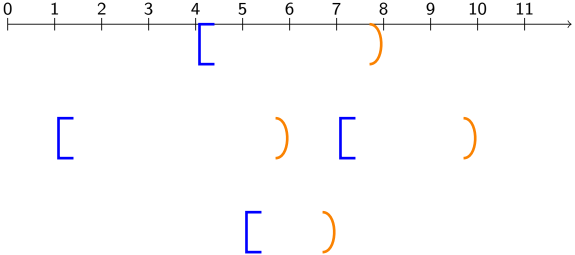
  - Maximum processing time p_{max}: $P||C_{max} \in$ FPT
- [Hermelin, Karhi, Pinedo, Shabtay 2018]
  - #distinct d_j, #distinct p_j, #distinct w_j: $1||\sum w_j U_j \in$ FPT with two of the three

Pathwidth μ



Definition: pathwidth

$$\mu \stackrel{\text{def}}{=} \max_{0 \le t < \max_j d_j} |\{j \mid r_j \le t < d_j\}| - 1$$

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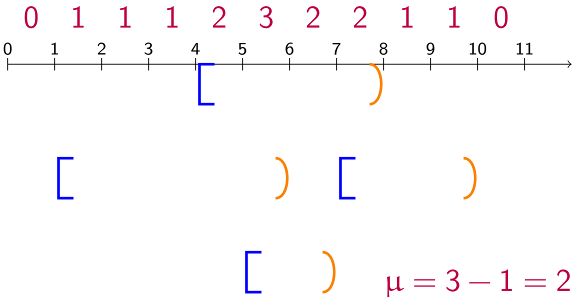

$\mu = 3 - 1 = 2$

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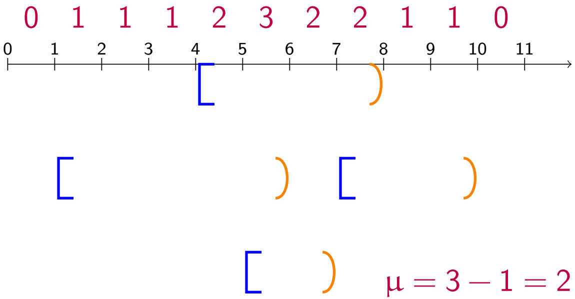

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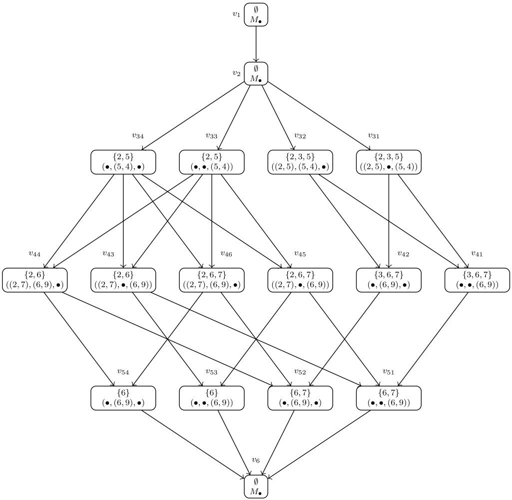
- = Pathwidth of the interval graph induced by the time windows
  - Nodes = time windows
  - there is an edge if they intersect

Pathwidth μ (cont.)

Recent work:

- $1|r_j, s_{jk}, reject, \tilde{d}_j| \sum_{j \notin R} v_j - \sum_{j \notin R} w_j T_j \in$ FPT (μ+slack) [de Weerdt et al. 2020]
- $P|prec, p_j = 1, r_j|L_{max} \in$ FPT (μ) [Munier-Kordon, Tang 2021]
- $P|M_j(type), prec, r_j|L_{max} \in$ FPT ($\mu + p_{max}$) [Hanen, Munier-Kordon 2022]

Pathwidth μ (cont.)



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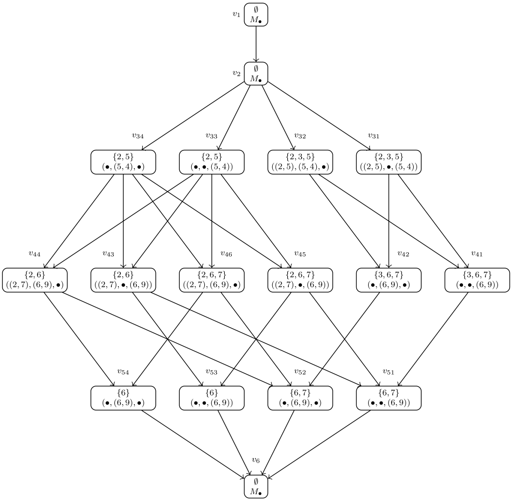

- FPT algorithm in time $\mathcal{O}(f(\mu) \times [n^4 + (\mu^{\mu+2} \times !\mu)n])$ with $f(\mu) = [(\mu+1)^{\mu^2} \times 2^\mu]^2$

$1|chains(l_{i,j}), p_j = 1, r_j, d_j|\star$ EXACT DELAYS

- Decision problem: $1|chains(l_{i,j}), p_j = 1, r_j, d_j|\star$
- EXACT DELAYS: $J_i \prec J_j \implies \sigma(j) = \sigma(i) + 1 + l_{i,j}$

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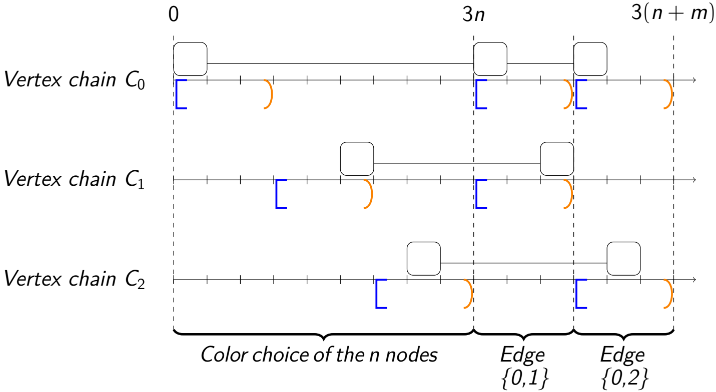
  - Example: k-COLORING, parameter $k = \#colors$
    - 3-COLORING is NP-hard

$1|chains(l_{i,j}), p_j = 1, r_j, d_j|\star$ EXACT DELAYS, pathwidth 1

- Reduction from 3-COLORING: $G = (V, E), n = |V|, m = |E|$

Example: $V = \{0, 1, 2\}, E = (\{0, 1\}, \{0, 2\})$



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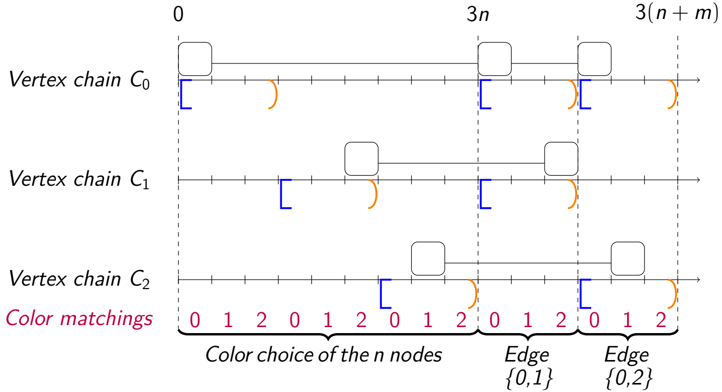

$P|chains(l_{i,j}), p_j = 1, r_j, d_j|\star$ EXACT DELAYS, max delay 1

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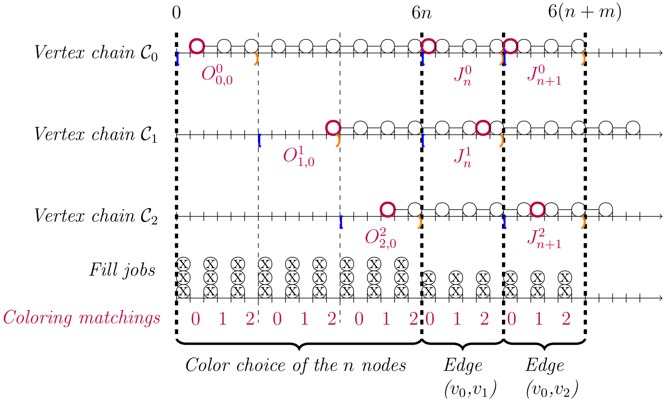
Example: $V = \{0, 1, 2\}, E = (\{0, 1\}, \{0, 2\})$, #machines $= n = 3$



Thank you!